International Journal of Theoretical Physics, Vol. 38, No. 11, 1999

Spacetime Metric and Lightcone Fluctuations

L. H. Ford[1]

Received March 9, 1999

Several aspects of the quantum fluctuations of spacetime geometry are discussed. A model for lightcone fluctuations is described in which a bath of gravitons leads to metric fluctuations. The operational definitions of such phenomena as lightcone and horizon fluctuations are examined. The problem of describing fluctuations of a quantum stress tensor is also discussed. The possibility that one can gain some insights about spacetime geometry fluctuations from studies of the force fluctuations on material bodies is suggested.

1. INTRODUCTION

The quantum nature of the gravitational field will necessarily lead to fluctuations of the spacetime geometry. Even an unquantized gravitational field coupled to a quantum matter field will experience fluctuations. Thus it is useful to draw a distinction between "active" and "passive" metric fluctuations. Active fluctuations are those in which gravity itself is quantized and the fluctuations are due to the dynamical degrees of freedom of the gravitational field. Passive fluctuations arise when gravity is coupled to a quantized matter field whose stress tensor is undergoing quantum fluctuations. Both types of metric fluctuations will cause fluctuations of the classical lightcone, and hence of horizons. Such fluctuations might be expected to produce a variety of physical phenomena. Pauli and others [1, 2] have speculated that lightcone fluctuations could act as a universal regulator to remove the divergences of quantum field theory. This hypothesis is yet to be either proven or disproven. One might also wonder whether or not lightcone fluctuations could invalidate the classical singularity theorems. These theorems depend crucially upon assumption of some type of energy condition on the matter-stress tensor and upon an analysis of the focusing properties of null rays.

[1] Institute of Cosmology, Department of Physics and Astronomy, Tufts University, Medford, Massachusetts 02155.

0020-7748/99/11 00-2941$16.00/ 0

The fact that the classical energy conditions are not fulfilled in general by the expectation value of a quantum stress tensor has been much discussed. However, the breakdown of the concept of focusing in quantum theory seems to have received little attention.

In this paper, we will not attempt to address either ultraviolet divergence or singularity avoidance, but rather describe some simple models within which one can attempt to understand better the physical meaning of metric and lightcone fluctuations. In Section 2, some of the effects of linearized quantum gravity and its active metric fluctuations are discussed. It is shown how these lead to lightcone fluctuations and to horizon fluctuations. Black hole horizon fluctuations even at the Planck scale pose a potentially serious challenge to the Hawking effect, as is discussed. However, it is argued that these fluctuations are strongly suppressed, and that their neglect in black hole evaporation is justified.

In Section 3, some issues associated with the passive metric fluctuations are discussed. In particular, one must address the problem of defining expectation values of products of stress tensors. Two possible approaches to this problem are considered. One of these defines such quantities as the squared energy density at a point through normal ordering. The other gives up the notion that such a quantity has any meaning, and attempts to deal only with finite integrals of stress tensor products. Some consequences of this latter approach are illustrated with a model of fluctuating electromagnetic forces on a material body. The paper concludes with a summary and discussion in Section 4.

2. A MODEL FOR ACTIVE METRIC FLUCTUATIONS

A complete treatment of active metric fluctuations would require a full quantum theory of gravity, which is not yet available. However, it is still possible to construct simplified models which one hopes reproduce some of the features expected in the more complete theory. One of these feature is quantum fluctuations of the lightcone. In this section, such a model will be discussed. Although perturbative quantum gravity is plagued with divergence problems, there is one level on which the notion of a quantized gravitational field is well defined. This is at the level of quantization of linearized perturbations of a given background spacetime. So long as one does not address questions about the interaction of gravitons with one another, no ambiguities arise. This theory is, however, sufficiently rich to contain nontrivial physics. The physical situation which it describes can be a bath of noninteracting gravitons on an arbitrary background spacetime. Here we wish to develop a formalism [3–5] which allows us to describe the effects of this bath upon the propagation of light rays on the background, and look for evidence of lightcone smearing.

Consider an arbitrary background metric $g^{(0)}_{\mu\nu}$ with a linear perturbation $h_{\mu\nu}$, so the spacetime metric is[2]

$$ds^2 = (g^{(0)}_{\mu\nu} + h_{\mu\nu})\, dx^\mu\, dx^\nu \tag{1}$$

For any pair of spacetime points x and x', let $\sigma(x, x')$ be one half of the squared geodesic separation in the full metric, and $\sigma_0(x, x')$ be the corresponding quantity in the background metric. We can expand $\sigma(x, x')$ in powers of $h_{\mu\nu}$ as

$$\sigma = \sigma_0 + \sigma_1 + \sigma_2 + \cdots \tag{2}$$

where σ_1 is first order in $h_{\mu\nu}$, etc. We now suppose that the linearized perturbation $h_{\mu\nu}$ is quantized, and that the quantum state $|\psi\rangle$ is a "vacuum" state in the sense that we can decompose $h_{\mu\nu}$ into positive- and negative-frequency parts $h^+_{\mu\nu}$ and $h^-_{\mu\nu}$, respectively, such that

$$h^+_{\mu\nu}|\psi\rangle = 0, \qquad \langle\psi|h^-_{\mu\nu} = 0 \tag{3}$$

It follows immediately that

$$\langle h_{\mu\nu}\rangle = 0 \tag{4}$$

in state $|\psi\rangle$. In general, however, $\langle (h_{\mu\nu})^2\rangle \neq 0$, where the expectation value is understood to be suitably renormalized. This reflects the quantum metric fluctuations.

2.1. Lightcone Fluctuations

We now wish to average the retarded Green's function, $G_{\rm ret}(x, x')$, for a massless field over the metric fluctuations. In a curved spacetime, $G_{\rm ret}(x, x')$ can be nonzero inside the future lightcone as a result of backscattering off of the spacetime curvature. However, its asymptotic form near the lightcone is the same as in flat spacetime:

$$G_{\rm ret}(x, x') \sim \frac{\theta(t - t')}{4\pi}\, \delta(\sigma), \qquad \sigma \to 0 \tag{5}$$

We will ignore the backscattered portion, and average this delta-function term over the fluctuations. Equation (5) can be expressed as

[2] Units in which $\hbar = c = 16\pi G = 1$ will be used in this paper. Thus the units of mass, length, and time differ by factors of $\sqrt{16\pi}$ from the usual definitions of the Planck mass, Planck length, and Planck time. The metric signature will be $(1, -1, -1, -1)$.

$$G_{\rm ret}(x, x') = \frac{\theta(t - t')}{8\pi^2} \int_{-\infty}^{\infty} d\alpha\, e^{i\alpha\sigma_0}\, e^{i\alpha\sigma_1} \tag{6}$$

We next use the relation

$$\langle e^{i\alpha\sigma_1} \rangle = e^{-\alpha^2 \langle \sigma_1^2 \rangle / 2} \tag{7}$$

Thus when we average over the metric fluctuations, the retarded Green's function is replaced by its quantum expectation value:

$$\langle G_{\rm ret}(x, x') \rangle = \frac{\theta(t - t')}{8\pi^2} \int_{-\infty}^{\infty} d\alpha\, e^{i\alpha\sigma_0}\, e^{-\alpha^2 \langle \sigma_1^2 \rangle / 2} \tag{8}$$

This integral converges if $\langle \sigma_1^2 \rangle > 0$ and can be evaluated to yield

$$\langle G_{\rm ret}(x, x') \rangle = \frac{\theta(t - t')}{8\pi^2} \sqrt{\frac{\pi}{2\langle \sigma_1^2 \rangle}}\, \exp\left(-\frac{\sigma_0^2}{2\langle \sigma_1^2 \rangle} \right) \tag{9}$$

Note that this averaged Green's function is indeed finite at $\sigma_0 = 0$ provided that $\langle \sigma_1^2 \rangle \neq 0$. Thus the lightcone singularity has been smeared out, as illustrated in Fig. 1. Note that the smearing occurs in both the timelike and spacelike directions.

We can find a general expression for $\langle \sigma_1^2 \rangle$. Let us first consider timelike geodesics, for which $ds^2 > 0$. Let $u^\mu = dx^\mu / d\tau$ be the tangent to the geodesic and τ be the proper time. We will define $\langle \sigma_1^2 \rangle$ by integrating along the unperturbed geodesic, in which case u^μ is normalized to unity in the background metric:

$$g^{(0)}_{\mu\nu} u^\mu u^\nu = 1 \tag{10}$$

The geodesic interval in the unperturbed metric is given by

$$\sigma_0 = \tfrac{1}{2} (\Delta\tau)^2 \tag{11}$$

where $\Delta\tau$ is the proper time elapsed along the geodesic. We have

$$\frac{ds}{d\tau} = \sqrt{1 + h_{\mu\nu} u^\mu u^\nu} \approx 1 + \frac{1}{2} h_{\mu\nu} u^\mu u^\nu \tag{12}$$

and hence the geodesic length between a pair of points in the perturbed metric is $\Delta s = \Delta\tau + \Delta s_1$, where

$$\Delta s_1 = \frac{1}{2} \int d\tau\, h_{\mu\nu} u^\mu u^\nu \tag{13}$$

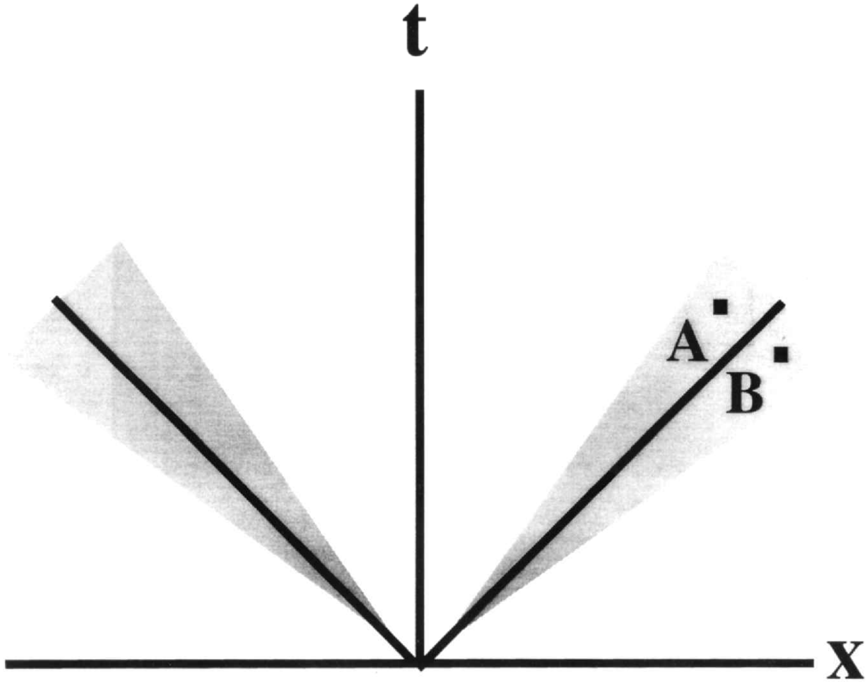

Fig. 1. The smearing of the lightcone due to metric fluctuations. A photon which arrives at point A from the origin has been slowed by the effect of metric fluctuations. A photon which arrives at point B has been boosted by metric fluctuations, and appears to travel at a superluminal velocity in the background metric.

Thus

$$\sigma = \frac{1}{2}(\Delta s)^2 = \frac{1}{2}(\Delta\tau)^2 + \Delta\tau\Delta s_1 + O(h^2) \tag{14}$$

and hence $\sigma_1 = \Delta\tau\Delta s_1$. If we average over the metric fluctuations, the result is

$$\langle\sigma_1^2\rangle = \frac{1}{2}\sigma_0 \int d\tau_1\, d\tau_2\, u_1^\mu u_1^\nu u_2^\rho u_2^\sigma \langle h_{\mu\nu}(x_1) h_{\rho\sigma}(x_2)\rangle \tag{15}$$

where $u_1^\mu = dx^\mu/d\tau_1$ and $u_2^\mu = dx^\mu/d\tau_2$. An analogous expression holds for the case of a spacelike geodesic, in which the integrations are over the proper length parameter of the geodesic:

$$\langle\sigma_1^2\rangle = -\frac{1}{2}\sigma_0 \int d\lambda_1\, d\lambda_2\, u_1^\mu u_1^\nu u_2^\rho u_2^\sigma \langle h_{\mu\nu}(x_1) h_{\rho\sigma}(x_2)\rangle \tag{16}$$

where now $u_1^\mu = dx^\mu/d\lambda_1$ is the tangent to the geodesic, and λ is the proper length. Here we have $\sigma_0 = -\frac{1}{2}(\Delta\lambda)^2$. Note that in Eqs. (15) and (16), the integration is along the mean trajectory of the photon, whereas the actual path in a fluctuating geometry (to the extent it has a meaning) is some stochastic path, as illustrated in Fig. 2.

The operational meaning of the smeared lightcone can be understood by considering a source and a detector of photons. If we ignore the finite sizes of photon wavepackets, then in the absence of lightcone fluctuations, all photons should traverse the interval between the source and the detector in the same amount of time. The effect of the lightcone fluctuations is to cause some photons to travel slower than the classical light speed and others to travel faster. The Gaussian function in Eq. (9) is symmetrical about the classical lightcone, $\sigma_0 = 0$, so the quantum lightcone fluctuations are equally likely to produce a time advance as a time delay, as illustrated in Fig. 1. The typical time delay or advance is of the order of

$$\Delta t \approx \frac{\sqrt{\langle\sigma_1^2\rangle}}{r} \tag{17}$$

where r is the distance between the source and detector. Note that Δt is an ensemble-averaged time variation. It is not necessarily the expected variation

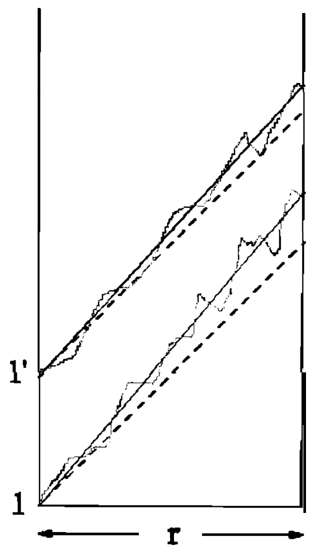

Fig. 2. A photon is emitted at point 1 and detected at point 2 at a distance r. A second photon is emitted at point 1′ and detected at point 2′. In the absence of metric fluctuations, a photon propagates on the classical lightcone, illustrated by the dashed lines. Metric fluctuations cause the photon to move on a stochastic trajectory with a propagation time which may either be larger or smaller than the classical flight time. The mean trajectory for a fixed flight time is illustrated by the solid lines.

in the arrival times of two photons which are emitted by the source in rapid succession. The reason for this is that the spacetime geometry fluctuates on a characteristic time scale of the order of the typical graviton wavelength. If the interval between the emission of the two photons is small compared to this timescale, they both travel in nearly the same spacetime geometry. If the interval is much longer, the geometry has changed significantly, and the variation in flight times is then Δt. This issue is discussed in detail in ref. 4.

Under most circumstances, the effects of the lightcone fluctuations will be exceedingly small. Consider, for example, a thermal bath of gravitons at temperature T. If the flight distance is large compared to the average graviton wavelength, then [3, 4]

$$\Delta t \approx \frac{1}{10} \sqrt{\frac{r}{T}} = \frac{1}{10} \ell_p \sqrt{\frac{rT_p}{\ell_p T}} \tag{18}$$

where ℓ_p and T_p are the Planck length and Planck temperature, respectively. Suppose, for example, that the universe is currently filled with a thermal bath of gravitons at $T = 3$ K and that r is the distance to a typical quasar, $r \approx 10^{10}$ cm. Then one finds that $\Delta t \approx 10^{-32}$ sec, presumably much too small to be detectable.

2.2. Black Hole Horizon Fluctuations

There is a particular situation in which lightcone fluctuations take on added interest. This is when the lightcone in question is a spacetime horizon, and we hence have horizon fluctuations. The possibility of horizon fluctuations takes on a special significance in the case of a black hole horizon. In addition to the possibility of information leaking out of the interior of a black hole into the exterior region, there is a chance that even minute horizon fluctuations could upset the beautiful connection between thermodynamics and black hole physics that was discovered by Hawking.

To understand the potential problem posed by horizon fluctuations, let us recall the essential features of Hawking's derivation, as given in the original paper [6]. Consider the spacetime of a black hole formed by gravitational collapse (Fig. 3). The null ray which forms the future horizon leaves $\mathcal{I}^-$ at advanced time $v = v_0$. The modes into which the outgoing thermal radiation will be created leave $\mathcal{I}^-$ at values of v slightly less than v_0, pass through the collapsing body, and reach $\mathcal{I}^+$ as outgoing rays, on which the retarded time u is constant. Hawking shows that the relation between the values of v and of u is (in our units where $16\pi G = 1$)

$$u = -\frac{M}{4\pi} \ln\left(\frac{v_0 - v}{A}\right) \tag{19}$$

where A is a constant. Thus $u \to \infty$ as $v \to v_0$. As seen by an observer at

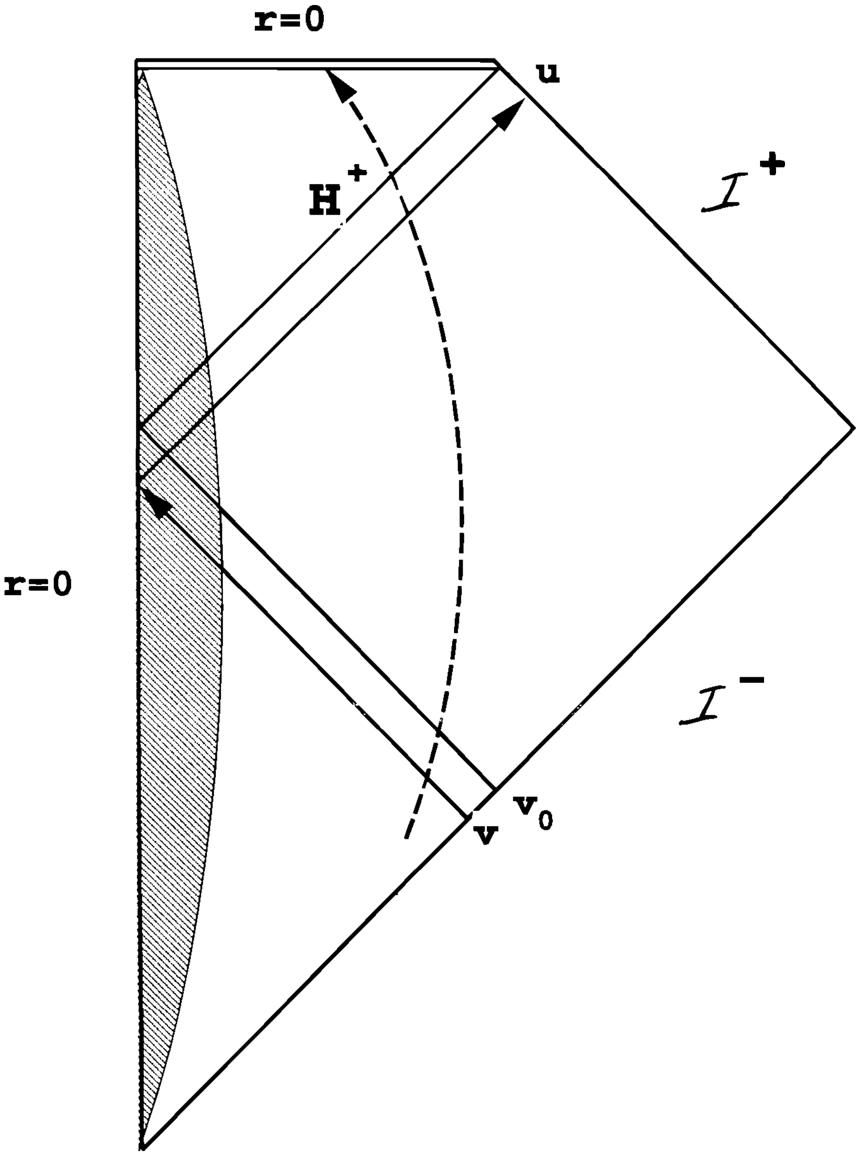

Fig. 3. The spacetime for a black hole formed by gravitational collapse. The shaded region is the interior of the collapsing star. A null ray which leaves $\mathscr{I}^-$ with advanced time v_0 becomes the future horizon, H^+. A ray which leaves at an earlier time v passes through the collapsing body and reaches $\mathscr{I}^+$ at retarded time u. The dashed line is the worldline of an observer who falls into the black hole after its formation.

infinity, these outgoing rays must hover extremely close to the horizon for a very long time. If one starts with a black hole with a mass M large compared to the Planck mass m_p, the semiclassical description should hold for the time required for the black hole to lose most of its original mass. Let

$$t_{\text{evap}} = M^3 = M\left(\frac{M}{m_p}\right)^2 \tag{20}$$

be this characteristic evaporation time. The basic problem posed by the horizon fluctuations is that they may cause an outgoing ray either to fall back into the black hole or else to prematurely escape. In either case, the semiclassical picture of black hole radiance would need to be modified at times less than t_{evap}.

In ref. 5 this question is analyzed using the lightcone smearing formalism described earlier. In order to do a detailed analysis, it would be necessary to obtain the renormalized graviton two-point function in a black hole spacetime in order to calculate $\langle\sigma_1^2\rangle$ using Eq. (15). This is would be a very difficult undertaking, so in ref. 5 a heuristic, order-of-magnitude analysis was given. The crucial assumption in this analysis was that the renormalized graviton two-point function near the horizon in the frame of reference of geodesic observers falling from rest at infinity is of the order of the inverse square of the horizon radius. This assumption is motivated by the observation that $\langle\phi^2\rangle$, where ϕ is a scalar field, typically is of the order of the inverse square of the local radius of curvature of spacetime when the quantum state in question is a "vacuum-like" state which does not depend upon any length scales other than those of the spacetime geometry. It is consistent with an estimate given by York [7], who quantized the lowest modes of vibration of a Schwarzschild black hole and found that the root-mean-square metric fluctuations in these modes is of the order of that given by this assumption.

It is necessary to have an operational definition of the magnitude of the horizon fluctuations. One such definition can be formulated by considering an infalling observer crossing the event horizon (see Fig. 4). If there is a fixed classical horizon, then all outgoing light rays emitted by this observer before some proper time τ_c will reach infinity, and none of the rays emitted after τ_c will do so. If, however, the horizon is undergoing quantum fluctuations, there will be a finite time interval peaked around τ_c in which rays may either escape to infinity or fall into the black hole. Thus some rays which are emitted after τ_c escape, and others emitted before τ_c, fall into the black hole. The horizon fluctuations will be the limit of lightcone fluctuations as the observer approaches the location of the classical horizon. Let $\Delta\tau_i$ be the uncertainty, due to lightcone fluctuations, in the time of emission of a ray which reaches a distant observer. When the infalling observer is very close

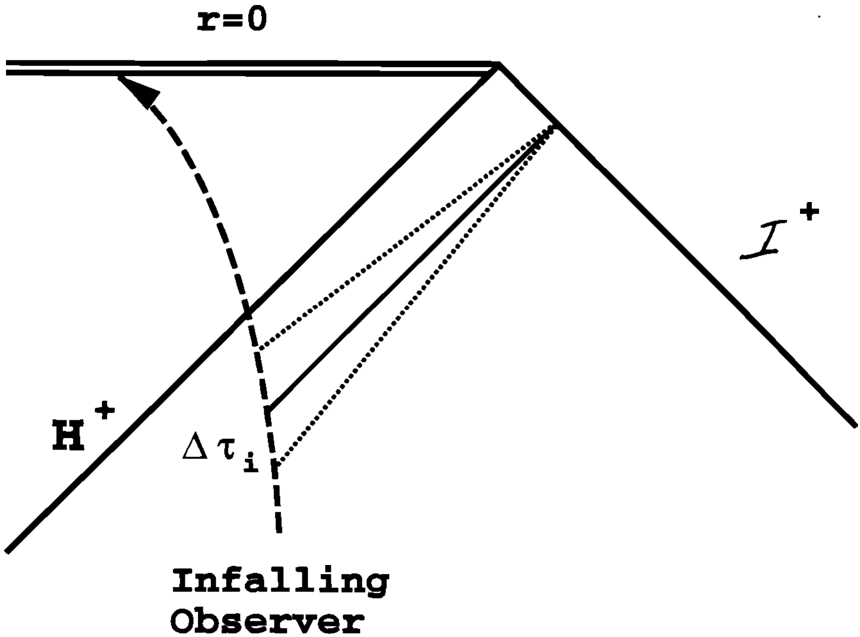

Fig. 4. An observer falling across the future horizon H^+ of a black hole emits photons which reach $\mathscr{I}^+$. In the presence of metric fluctuations, these photons need not follow the classical lightcone (solid line), but rather may follow timelike or spacelike paths in the background geometry (dotted lines). The characteristic variation in emission time, as measured in the frame of the infalling observer, of photons which reach $\mathscr{I}^+$ at the same point is $\Delta\tau_i$.

to the classical horizon, we wish to compare $\Delta\tau_i$ with the proper time required to cross the outgoing null lines associated with the modes which give the dominant contribution to the black hole evaporation. These outgoing line have $u \approx t_{\text{evap}}$, corresponding to a ray which enters the collapsing body just before black hole formation and reaches a large distance from the black hole late in the evaporation process. It may be shown that the proper time required to fall in the Schwarzschild geometry from $u \approx t_{\text{evap}}$ to $u = \infty$ is of the order of

$$\delta\tau \approx M\, e^{-M^2/m_p^2} \tag{21}$$

where M is the black hole mass and m_p is the Planck mass. Note that $\delta\tau$ is typically far below the Planck length, so even Planck-scale fluctuations could greatly alter black hole thermodynamics. It is shown in ref. 5 that as the observer crosses the $u \approx t_{\text{evap}}$ ray, the uncertainty in the time of emission of an outgoing ray is of the order of

$$\Delta\tau_i \approx \frac{m_p}{M}\,\delta\tau \tag{22}$$

Hence so long as $M \gg m_p$, $\Delta\tau_i \ll \delta\tau$. From this result, we conclude that the horizon fluctuations do not invalidate the semiclassical derivation of the Hawking effect until the black hole's mass approaches the Planck mass. This is the point at which we would expect the semiclassical treatment to fail.

3. PASSIVE METRIC FLUCTUATIONS

We now turn to the problem of spacetime metric fluctuations which are driven by quantum fluctuations of the source, the matter-stress tensor. The expectation value of the stress tensor operator for a quantum field is formally divergent and needs to be renormalized. In Minkowski spacetime, this may be accomplished simply by normal ordering with respect to the Minkowski vacuum state. This amounts to defining the stress tensor in the vacuum state to be zero. On a curved spacetime, the renormalization is more complicated, but has been throughly investigated. In order to address the fluctuations of the stress tensor, it is necessary to give meaning to the expectation value of a product of stress tensor operators. For example, we would say the the energy density at a point is fluctuating significantly when the expectation value of the squared energy density at that point differs noticeably from the square of the expectation value.

We will restrict our attention here to the case of Minkowski spacetime. Even here the problem of defining squared stress tensors is nontrivial. Let $T(x) = :\phi_1(x)\phi_2(x):$ be a normal-ordered quadratic operator, such as a stress tensor component. The expectation value of T in any physically realizable state is finite. In Minkowski spacetime, normal ordering simply means subtraction of the expectation value in the Minkowski vacuum state:

$$T(x) = :\phi_1(x)\phi_2(x) := \phi_1(x)\phi_2(x) - \langle\phi_1(x)\phi_2(x)\rangle_0 \tag{23}$$

Now consider the square of T. It may be shown using Wick's theorem that

$$T(x)T(x') = S_0 + S_1 + S_2 \tag{24}$$

where

$$S_0 = \langle\phi_1(x)\phi_1(x')\rangle_0; \langle\phi_2(x)\phi_2(x')\rangle_0 + \langle\phi_1(x)\phi_2(x')\rangle_0 \langle\phi_2(x)\phi_1(x')\rangle_0 \tag{25}$$

$$\begin{aligned} S_1 = {} & :\phi_1(x)\phi_1(x'): \langle\phi_2(x)\phi_2(x')\rangle_0 + :\phi_1(x)\phi_2(x'): \langle\phi_2(x)\phi_1(x')\rangle_0 \\ & + :\phi_2(x)\phi_1(x'): \langle\phi_1(x)\phi_2(x')\rangle_0 + :\phi_1(x)\phi_2(x'): \langle\phi_2(x)\phi_1(x')\rangle_0 \end{aligned} \tag{26}$$

and

$$S_2 = :\phi_1(x)\phi_2(x)\phi_1(x')\phi_2(x'): \tag{27}$$

Thus the operator product $T(x)T(x')$ consists of a purely vacuum part S_0, a fully normal-ordered part S_2, and a part S_1 which is a cross term between the vacuum and normal-ordered parts.

So long as x and x' are distinct nonnull-separated points, all three parts have finite expectation values. However, in the coincidence limit, $x' = x$, only the fully normal-ordered part remains finite. The purely vacuum part does not pose a serious problem, as we can restrict our attention to the difference in the expectation value in an arbitrary state and in the vacuum state:

$$\langle T(x)T(x')\rangle - \langle T(x)T(x')\rangle_0 = \langle S_1\rangle + \langle S_2\rangle \tag{28}$$

Although $\langle S_2\rangle$ is always finite, $\langle S_1\rangle$ is both infinite and state dependent in the coincidence limit.

There seem to be two possible approaches to this problem. One is to impose some additional renormalization to remove the infinity, and the other is to replace the local operator products by finite spatially or temporally averaged quantities. If one adopts the former approach, the simplest possibility is to drop the cross term S_1 and use only the fully normal-ordered part. This approach was used in refs. 8 and 9, where it was shown that one obtains the correct classical limit in the sense that

$$\langle :T(x)T(x'):\rangle = \langle S_2\rangle = \langle T(x)\rangle\, \langle T(x')\rangle \tag{29}$$

if the quantum state is a coherent state. This implies that a classical field excitation, which is described by a coherent state, exhibits no quantum fluctuations in its stress tensor.

The fluctuations of the energy density and the limits of the semiclassical theory of gravity were examined in ref. 9 using the normal-ordered definition of the squared energy density. It was found that the fractional fluctuations in the energy density, as measured by the quantity

$$\Delta = \frac{\langle \rho^2\rangle - \langle \rho\rangle^2}{\langle \rho^2\rangle} \tag{30}$$

become large for states with locally negative energy density. These include squeezed states and Casimir vacuum states. For the case of the Casimir effect for a massless scalar field, one can prove that $\Delta \geq 1/3$, and typically one finds from explicit calculation that Δ is of order unity. Thus if one were to measure the local Casimir energy density at a given point in space and time, the result of a given measurement is likely to vary by a factor of two or more from the mean value. This implies that the gravitational field created by a highly squeezed state or by Casimir energy is not described by a fixed classical metric, but rather by a fluctuating metric. The flat spacetime results of ref. 9 have been generalized to symmetrical curved spacetimes by Phillips and Hu [10].

An important area of application of metric fluctuations is to the study of the early universe, especially to the problem of structure formation. This topic has been discussed by many authors (see, for example, refs. 11 and 12). In many models, quantum fluctuations of a matter field produce primordial density perturbations that later form galaxies. This is one example where small fluctuations have the potential to eventually produce large effects.

One may apply the normal-ordering prescription to study fluctuations in the Hawking flux from an evaporating black hole. Let $F(t) = T_{rt}$ be the flux operator. The correlation function

$$C(t, t') = \langle :F(t)F(t'): \rangle - \langle F(t) \rangle \langle F(t') \rangle \tag{31}$$

has a value of the order of $\langle F \rangle^2$ when $t = t'$, and decays monotonically with increasing $|t - t'|$ with a characteristic time scale of the order of M, the light travel time across the black hole [15]. This implies that the Hawking flux undergoes large fluctuations on a time scale of M.

If one wishes to retain the S_1 term, there seems to be no alternative but to give up the notion of a well-defined local energy density. It is still possible to have finite observables, provided that they can be expressed as convergent integrals. Whether this can always be done is unclear. Barton [13] advocated this approach for the study of the fluctuations of the Casimir force. He examined a force operator which has been averaged over space and time and found that the fluctuations of this operator are finite for a nonzero averaging time, but diverge in the limit that this time goes to zero. When the stress tensor is exerting a force on a material body, there is a natural cutoff provided by the fact that real materials are transparent to very high frequency modes. In this case, the infinite fluctuations can be explained away as an artifact of an unphysical assumption of perfect reflectivity. No such cutoff seems to exist in gravity, short of the Planck scale. High-frequency modes produce increasing gravitational effects as the frequency rises. There is a possibility that exotic physics at the Planck scale introduces a cutoff through a fractal nature of spacetime at that scale. This idea is, however, highly speculative. In any case, a cutoff at the Planck scale would not prevent unacceptably large effects from arising.

This point may be illustrated by considering linearized gravity coupled to a quantum matter field. The metric perturbation due to a classical source $T_{\mu\nu}$ is, in the harmonic gauge,

$$h_{\mu\nu}(x) = \frac{1}{2} \int d^4x' \; G_{\mathrm{ret}}(x - x') T_{\mu\nu}(x') \tag{32}$$

Now let $T_{\mu\nu}$ and hence $h_{\mu\nu}$ become quantum operators. The correlation function for the metric perturbation becomes

$$\langle h_{\mu\nu}(x) h_{\rho\sigma}(x')\rangle = \frac{1}{4}\int d^4x_1\, d^4x_2\, G_{\rm ret}(x - x_1) G_{\rm ret}(x' - x_2) \times \langle :T_{\mu\nu}(x_1): :T_{\rho\sigma}(x_2):\rangle \tag{33}$$

We might try to use this correlation function to calculate the rate at which gravitons are emitted by a cavity containing photons is some quantum state. If only the fully normal-ordered part of the stress tensor product is retained, this rate is finite and usually very small [8, 14]. However, if we keep the S_1 cross terms, the integrand is singular at points where $x_1 = x_2$. The leading term will go as

$$\frac{1}{(x_1 - x_2)^6}, \qquad x_2 \to x_1 \tag{34}$$

and appears to render the integral for the power radiated in gravitons infinite. If one adopts the view that Planck-scale physics is needed to avoid this infinity, one would presumably cut off the integration at the point that $|x_1 - x_2| \approx \ell_p$. This leads to a very large answer, at least of the order of E/ℓ_p, where E is the electromagnetic energy in the cavity, in which case all of this energy would be radiated in gravitons on the order of a Planck time. The observed fact that microwave ovens function without noticeable loss of power to gravitons proves that this line of argument is false.

An alternative approach might be to try to redefine these apparently divergent integrals by an integration by parts. The basic idea can be illustrated as follows:

$$\begin{aligned}
&\int_{-\infty}^{\infty} dt_1\, dt_2\, f(t_1) f(t_2) \frac{1}{(t_1 - t_2)^4} \\
&\quad = -\frac{1}{12}\int_{-\infty}^{\infty} dt_1\, dt_2\, f(t_1) f(t_2) \frac{\partial^4}{\partial t_1^2 \partial t_2^2} \ln[(t_1 - t_2)^2 \mu^2] \\
&\quad = -\frac{1}{12}\int_{-\infty}^{\infty} dt_1\, dt_2\, \ddot{f}(t_1) \ddot{f}(t_2) \ln[(t_1 - t_2)^2 \mu^2]
\end{aligned} \tag{35}$$

where μ is an arbitrary constant. We have assumed that the function $f(t)$ vanishes as $|t| \to \infty$, so the surface terms in the integration by parts vanish. The effect of this manipulation is to replace the apparently nonintegrable singularity in the first integral by a mild, integrable singularity in the final integral. This trick has been employed by various authors under the labels "generalized principal value integration" [16] or "differential regularization" [17].

Let us discuss briefly a simple model in which this idea may be applied. Consider an electrically polarizable particle (e.g., an atom) which interacts with the electromagnetic field through the interaction Hamiltonian

$$H_{\rm int} = -\frac{1}{2}\alpha :\mathbf{E}^2: \tag{36}$$

where $\mathbf{E}$ is the quantized electric field operator and α is the polarizability, which will here assumed to be independent of frequency. If the particle has mass m and moves nonrelativistically, then the force operator is

$$\mathbf{F} = -\nabla H_{\rm int} \tag{37}$$

and the velocity operator is

$$\mathbf{v}(t) = \mathbf{v}_i + \frac{1}{m}\int_{t_i}^{t} dt'\ \mathbf{F}(t') = \mathbf{v}_i - \frac{\alpha}{2m}\int_{t_i}^{t} dt'\ \nabla :\mathbf{E}^2: \tag{38}$$

where $\mathbf{v}_i$ is the velocity at time $t = t_i$. The quantum fluctuations of the electromagnetic field result in a fluctuating force acting upon the particle. This in turn causes it to undergo Brownian motion, for which Eq. (38) may be viewed as being the Langevin equation. The mean classical trajectory of the particle is described by

$$\langle \mathbf{x} \rangle = \mathbf{x}_i + \int_{t_i}^{t} dt'\ \langle \mathbf{v}(t') \rangle \tag{39}$$

and the fluctuations around this trajectory are described by quantities such as $\langle \mathbf{v}^2 \rangle$.

Consider the case of a particle which starts from rest in the distant past, so $\mathbf{v}_i = 0$ and $t_i = -\infty$. Let the quantum state of the electromagnetic field be a coherent state for a single mode. In this case, the velocity fluctuations come entirely from the S_1 cross term, as can be seen from Eq. (29). From Eqs. (26) and (38) we find that

$$\Delta\langle \mathbf{v}^2 \rangle = \langle \mathbf{v}^2 \rangle - \langle \mathbf{v} \rangle \cdot \langle \mathbf{v} \rangle = \frac{\alpha^2}{m^2}\int_{-\infty}^{t} dt_1\, dt_2\, \nabla_1 \cdot \nabla_2 [\langle :E_{1i} E_{2j}: \rangle \langle E_1^i\, E_2^j \rangle_0] \tag{40}$$

Here $\langle E_1^i\ E_2^j \rangle_0$ is the electric field two-point function, and the labels 1 and 2 refer to the spacetime location of the particle at times t_1 and t_2, respectively. Because the quantum state is a single mode coherent state,

$$\langle :E_{1i}\ E_{2j}: \rangle = \mathcal{E}_{1i} \mathcal{E}_{2j} \tag{41}$$

where $\mathcal{E}_i$ is the classical electric field in this state. In general, the integrand in Eq. (40) has to be evaluated along a mean classical trajectory of the particle. The problem is greatly simplified if we assume that the particle does not move appreciably in space and is at the origin of our coordinates, so after the spatial derivatives have been evaluated we set $\mathbf{x}_1 = \mathbf{x}_2 = 0$.

Take an explicit example in which the classical field is that of a monochromatic traveling wave:

$$\mathscr{E}_x = A\cos[\omega(t - z)], \qquad \mathscr{E}_y = \mathscr{E}_z = 0 \tag{42}$$

In this case, the relevant component of the two-point function is

$$\langle E_1^x E_2^x \rangle_0 = \frac{(\Delta t)^2 + (\Delta \mathbf{x})^2 - 2(\Delta x)^2}{\pi^2[(\Delta t)^2 - (\Delta \mathbf{x})^2]^3} \tag{43}$$

where $\Delta \mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1$ is the spatial separation of the two points, and Δx is its x component. If we insert these forms into Eq. (40), the resulting integral is formally divergent. Furthermore, integration by parts alone does not solve the problem because there can be singular surface terms. Such terms can only be avoided if we assume that the classical electromagnetic field is adiabatically switched on and then off again, and we attempt to evaluate $\Delta\langle \mathbf{v}^2 \rangle$ only in the asymptotic region after the switchoff. If the classical field is switched on for a time of the order of $T \gg \omega^{-1}$, then a detailed calculation using results such as Eq. (35) yields

$$\Delta\langle \mathbf{v}^2 \rangle \approx \frac{2\alpha^2\omega^5 A^2}{15\pi m^2} T \tag{44}$$

Thus the particle undergoes Brownian motion with mean squared velocity increasing linearly in time. If one evaluates the magnitude of this effect for a real atom, the result is far too small to be experimentally observed. For example, a hydrogen atom placed in a laser beam of wavelength 6000 Å and intensity 10^6 W/cm^2 for 1 day would experience a $\Delta\langle \mathbf{v}^2 \rangle$ corresponding to a temperature of only 10^{-4} K.

4. SUMMARY AND DISCUSSION

We have examined simple models for both active and passive metric fluctuations. In the former case, it is possible to have lightcone fluctuations due to a bath of gravitons. Such fluctuations manifest themselves in a variable speed of propagation of photons through the spacetime region containing the gravitons. If spacetime is not Minkowskian, then even in the vacuum state there will be nonzero lightcone fluctuations. For example, a flat spacetime with periodicity in one spatial direction will exhibit lightcone fluctuations due to Casimir fluctuations of the quantized gravitational field [18]. The same effect occurs in curved spacetimes. In spacetimes with classical horizons, the metric fluctuations lead to quantum horizon fluctuations. For a black hole this has the possibility to greatly alter the evaporation process. However, estimates of the magnitude of the effect indicate that this does not in fact happen.

In the case of passive fluctuations due to a quantum stress tensor, the major challenge is to understand clearly stress tensor fluctuations. One can adopt the viewpoint that the expectation value of the squared energy density should be as well defined as a local quantity as is that of the energy density. In this case, a renormalization prescription such as normal ordering is needed. This approach does seem to give consistent results in that highly squeezed quantum states exhibit large stress tensor fluctuations, whereas coherent states do not. The other approach is to give up the notion of a local mean squared energy density, and look only at integrated quantities. If this viewpoint is correct, it must be possible to give a finite value to all integrals which represent observable quantities. Whether this can be done is not yet clear. The model of a polarizable particle in an electric field discussed in the previous section offers some insights into this problem. It is in fact possible to define the change in the mean squared velocity of the particle if one makes measurements only after the classical electric field has been switched off. Then the formally divergent integral can be redefined to yield a finite and rather small value. However, the notion of the instantaneous squared velocity seems to cease to have any meaning.

One may hope that this type of model can lead to insights for the case of gravity. It is reasonable to suppose that the same principles which determine the force fluctuations on material bodies should also determine the fluctuations in the source of the gravitational field. Although these force fluctuations may be very small, they are far larger than most quantum effects associated with gravity. This suggests the intriguing possibility that one might find a way to perform experiments on the former and thus learn about the latter.

ACKNOWLEDGMENTS

I thank C. H. Wu and H. Yu for helpful discussions. This work was supported in part by the National Science Foundation under Grant PHY-98 00965.

REFERENCES

[1] W. Pauli, *Helv. Phys. Acta. Suppl.* **4**, 69 (1956).
[2] S. Deser, *Rev. Mod Phys.* **29**, 417 (1957); B. S. DeWitt, *Phys. Rev. Lett.* **13**, 114 (1964).
[3] L. H. Ford, *Phys. Rev. D* **51**, 1692 (1995), gr-qc/941 0047.
[4] L. H. Ford and N. F. Svaiter, *Phys. Rev. D* **54**, 2640 (1996), gr qc/96 04 052.

[5] L. H. Ford and N. F. Svaiter, *Phys. Rev. D* **56**, 2226 (1997), gr-qc/9704050.
[6] S. W. Hawking, *Commun. Math. Phys.* **43**, 199 (1975).
[7] J. W. York, *Phys. Rev. D* **28**, 2929 (1983).
[8] L. H. Ford, *Ann. Phys.* (NY) **144**, 238 (1982).
[9] C.-I. Kuo and L. H. Ford, *Phys. Rev. D* **47**, 4510 (1993), gr-qc/9304008.
[10] N. G. Phillips and B. L. Hu, *Phys. Rev. D* **55**, 6123 (1997), gr-qc/9611012.
[11] E. Calzetta and B. L. Hu, *Phys. Rev. D* **52**, 6070 (1995), gr-qc//9505046.
[12] E. Calzetta, A. Campos, and E. Verdaguer, *Phys. Rev. D* **56**, 2163 (1997), gr-qc/9704010.
[13] G. Barton, *J. Phys. A* **24**, 991 (1991); **24**, 5563 (1991).
[14] S. del Campo and L. H. Ford, *Phys. Rev. D* **38**, 3657 (1988).
[15] C. H. Wu and L. H. Ford, *Phys. Rev. D* 6, 104013 (1999), gr-qc/990512.
[16] K. T. R. Davies and R. W. Davies, *Can. J. Phys.* **67**, 759 (1989); K. T. R. Davies, R. W. Davies, and G. D. White, *J. Math. Phys.* **31**, 1356 (1990).
[17] D. Z. Freedman, K. Johnson, and J. I. Latorre, *Nucl. Phys. B* **371**, 353 (1992).
[18] H. Yu and L. H. Ford, *Phys. Rev. D* **60**, 084023 (1999), gr-qc/9904082.